KINEMATIC AND CROFTON FORMULAS FOR LINEAR GROUPS.

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ABSTRACT. Let $G_q(\mathbf{R}^n)$ be the Grassmannian of all linear q dimensional subspaces of $\mathbf{R}^n$ and I an integral invariant of $p+q-n$ dimensional submanifolds of $\mathbf{R}^n$. Then we give methods of evaluating Crofton type integral

$$\int_{G_q(\mathbf{R}^n)} I(M \cap L)\, dL.$$

The methods also work for various generalizations of $G_q(\mathbf{R}^n)$ such as complex Grassmannians.

1. INTRODUCTION

In Section 4(d) of his paper [2], Griffiths proposes studying Crofton type formulas for curvature integrals of intersections of a submanifolds of $\mathbf{R}^n$ with the set of all linear, rather than the larger set of affine, subspaces of $\mathbf{R}^n$. To be more specific, let $G_q(\mathbf{R}^n)$ be the Grassmannian manifold of all q dimensional linear subspaces of $\mathbf{R}^n$ and M a compact p dimensional submanifold of $\mathbf{R}^n$. Then the problem is to evaluate integrals of the form

$$\int_{G_q(\mathbf{R}^n)} I(M \cap L)\, dL$$

where I is an "integral invariant" of $p+q-n$ dimensional submanifolds of $\mathbf{R}^n$ and dL is the $SO(n)$ invariant measure on $G_q(\mathbf{R}^n)$. In this note generalize by letting H be a closed subgroup of $O(n)$ which is transitive on the unit sphere S^{n-1} and L_0 a linear subspace of $\mathbf{R}^n$ and replacing $G_q(\mathbf{R}^n)$ with the manifold

$$\mathrm{Gr}(H, L_0) = \{aL_0 : a \in H\}.$$

For example if n is even and $\mathbf{R}^n$ is identified with $\mathbf{C}^{n/2}$, $H = U(/n/2)$ is the unitary group and L_0 is a complex linear subspace of $\mathbf{C}^{n/2} = \mathbf{R}^n$ of complex dimension q, then $\mathrm{Gr}(L_0, H)$ is the usual Grassmannian of all complex q dimensional linear subspaces of $\mathbf{C}^{n/2}$ with its measure invariant under $U(n/2)$.

In this note we use the methods of our earlier paper [3] to reduce the general case to concrete integrals over groups. While this is progress in the

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general case, getting the precise constants for a specific choice of H, L_0, and integral invariant I has to be done case by case.

2. Preliminaries

I am going to put the problem of a kinematic formula for intersections with a plane constrained to go through the origin in a more general setting. Let $H \subset O(n)$ be a closed subgroup that is transitive on the unit sphere S^{n-1}. The most natural example of such a group is $H = O(n)$ or $H = SO(n)$. If n is even and $\mathbf{R}^n$ is identified with $\mathbf{C}^{n/2}$ then $H = U(n/2)$ and $H = SU(n/2)$ are also examples that arise naturally. (More generally all the subgroups of $O(n)$ that act transitively on S^{n-1} have been classified [4, 1].) Let $\mathbb{R}^+$ be the multiplicative group of positive real numbers. Let $G = H \times \mathbb{R}^+$. The group G acts on $\mathbf{R}^n \smallsetminus \{0\}$ by

$$(g, r) \cdot x = rgx.$$

This is transitive on $\mathbf{R}^n \smallsetminus \{0\}$ because H is transitive on S^{n-1}.

Denote by $\mathsf{g}(\,,)$ the standard flat Riemannian metric on $\mathbf{R}^n$:

$$\mathsf{g} = \sum_{i=1}^{n} dx_i{}^2$$

and let $\langle\,,\rangle$ be the Riemannian metric defined by

$$\langle X, Y\rangle_x = \frac{\mathsf{g}_x(X,Y)}{|x|^2}.$$

The following is easy to verify.

Proposition 1. *The metric $\langle\,,\rangle$ on $\mathbf{R}^n \smallsetminus \{0\}$ is invariant under $O(n) \times \mathbf{R}^+$ and therefore also under its subgroup $G = H \times R^+$.* □

The group $O(n)$ has a natural (up to a constant multiple) bi-invariant Riemannian. If we use the usual identifications of the Lie algebra of $O(n)$ with both the left invariant vector fields on the group and with the skew-symmetric matrices then this metric is $(A, B)_{O(n)} \mapsto \frac{1}{2}\operatorname{tr}(A^t B)$ (where A^t is the transpose of A). Let $\mathbf{o} \in S^{n-1}$ be the north pole of the sphere S^{n-1} (which we define as $\mathbf{o} := e_n$ where $e_1, e_2, \ldots, e_n$ is standard orthonormal basis of $\mathbf{R}^n$). Then the natural map $\pi(g) = g(\mathbf{o})$ is a Riemannian submersion. (Requiring this to hold forces the factor of $\frac{1}{2}$ in the definition of the bi-invariant $(A, B)_{O(n)}$.) Use the metric on $H \subseteq O(n)$ induced by $(\,,)_{O(n)}$. Then the map $\pi\big|_H(g) = g(\mathbf{o})$ is also a Riemannian submersion. Giving $G = H \times \mathbf{R}^+$ the product metric we then have

Proposition 2. *If G has the product metric $(\,,)_{O(n)} + dr^2/r^2$, then this metric is bi-invariant and the map $\mathsf{p}\colon G \to \mathbf{R}^n \smallsetminus \{0\}$ is a given by $\mathsf{p}(g, r)(\mathbf{o}) = rg(\mathbf{o})$ is a Riemannian submersion. Letting Ω_G be the volume form of this metric, we also have that Ω_G is bi-invariant on G and G is unimodular as the volume form is bi-invariant.* □

Let M be a smooth compact submanifold of $\mathbf{R}^n$, possibly with boundary. For a function $f\colon M \to \mathbf{R}$ we have two notions of a measure on M:

$$\int_M p(x)\, dx = \text{integral of } p \textit{ w.r.t.} \text{ the } g\text{-volume on } M,$$

$$\int_M p(x)\, \Omega_M(x) = \text{integral of } p \textit{ w.r.t.} \text{ the } \langle\,,\rangle\text{-volume on } M.$$

Elementary calculations show these are related by

$$\int_M p(x)\, dx = \int_M |x|^k p(x)\, \Omega_M(x), \qquad \int_M p(x)\, \Omega_M(x) = \int_M \frac{p(x)}{|x|^k}\, dx.$$

In these formulas $|x|$ is computed with respect to the Euclidean metric g.

Let $\mathcal{P}$ be an invariant polynomial defined on the second fundamental forms of $p+q-n$ dimensional submanifolds of $\mathbf{R}^n$ (see [3, Def. 6.3, p. 40] for the definition). Define an integral invariant $I^{\mathcal{P}}$ on $p+q-n$ dimensional submanifolds of $\mathbf{R}^n$ by

$$I^{\mathcal{P}}(M) = \int_M \mathcal{P}(h^M_x)\, dx$$

where h^M_x is the vector valued second fundamental form of M at x with respect to the usual Euclidean inner product g.

Let M^p and N^q be submanifolds of $\mathbf{R}^n \setminus \{0\}$. Then our goal is to evaluate the integral

$$\int_G I^{\mathcal{P}}(M \cap gN)\, \Omega_G(dg). \tag{1}$$

Note this is not the usual set up for integral geometry as $I^{\mathcal{P}}$ is not invariant with respect to the action of the group G. To see how this relates to the problem of a kinematic formula for intersections with q-dimensional planes through the origin, let L_0 be some q-dimensional linear subspace of $\mathbf{R}^n$, and set

$$\mathrm{Gr}(L_0, G) = \{gL_0 : g \in G\} = \{aL_0 : a \in H\}. \tag{2}$$

If $H = O(n)$ then $\mathrm{Gr}(L_0, G)$ is the usual Grassmannian of all q-dimensional linear subspaces of $\mathbf{R}^n$. If $H = U(n/2)$ and L_0 is a complex linear subspace of $\mathbf{R}^n$ then $\mathrm{Gr}(L_0, G)$ is the complex Grassmannian of all $q/2$-dimensional complex subspaces of $\mathbb{C}^{n/2} = \mathbf{R}^n$. Note that $\mathrm{Gr}(L_0, G)$ has an H invariant volume element dL. (This is because H is compact.) This volume element is also invariant under the action of G (as the factor $\mathbb{R}^+$ acts trivially on $\mathrm{Gr}(L_0, G)$.) Then the original problem coming out of the Griffiths paper is to evaluate the integral

$$\int_{\mathrm{Gr}(L_0,G)} I^{\mathcal{P}}(M \cap L)\, dL = \int_H I^{\mathcal{P}}(M \cap aL_0)\, \Omega_H(da) \tag{3}$$

where Ω_H is the invariant volume form on H. But if we choose N to be a bounded open subset of L_0 with compact closure in $L_0 \setminus \{0\}$ then it is not hard to show there is a constant C, only depending on the $\langle\,,\rangle$ volume of N so that

$$\int_{\mathrm{Gr}(L_0,G)} I^{\mathcal{P}}(M\cap L)\,dL = C\int_G I^{\mathcal{P}}(M\cap gN)\,dg. \tag{4}$$

(This follows just as in [3, appendix to Section 3].) Thus if we can evaluate (1) then we can also evaluate (2). Let $\mathbf{o}$ be the north pole of S^{n-1} and let K be the isotropy subgroup of H at $\mathbf{o}$. Then K is also the isotropy subgroup of G acting on $\mathbf{R}^n \setminus \{0\}$ at the point $\mathbf{o}$. Thus $\mathbf{R}^n\setminus\{0\} = G/K$ is a homogeneous space.

As submanifolds of $\mathbf{R}^n\setminus\{0\}$ have two notions of surface area (coming from the two different Riemannian metrics g and $\langle\,,\rangle$) we need some notation to keep them straight. Let M^k be a k-dimensional submanifold of $\mathbf{R}^n\setminus\{0\}$ and $p\colon M\to\mathbf{R}$ a smooth function. Then

$$\int_M p(x)\,dx = \text{integral of } p \textit{ w.r.t.} \text{ the } \mathsf{g}\text{-volume on } M,$$

$$\int_M p(x)\,\Omega_M(dx) = \text{integral of } p \textit{ w.r.t.} \text{ the } \langle\,,\rangle\text{-volume on } M.$$

It follows from the definitions that these are related by

$$\int_M p(x)\,dx = \int_M |x|^k p(x)\,\Omega_M(dx), \quad \int_M p(x)\,\Omega_M(dx) = \int_M \frac{p(x)}{|x|^k}\,dx. \tag{5}$$

Let K be the isotropy subgroup of G at $\mathbf{o}$:

$$K := \{a\in G : a(\mathbf{o}) = \mathbf{o}\}.$$

Recalling that $\mathsf{p}\colon G\to\mathbf{R}^n\setminus\{0\}$ given by $\mathsf{p}(g,r) = rg(\mathbf{o})$ is a Riemannian submersion and each fiber $\mathsf{p}^{-1}[x]$ is a translate in G of K and thus has the same volume as K. Also p is a Riemannian submersion and thus of M is a k dimensional submanifold of $\mathbf{R}^n\setminus\{0\}$, $\widehat{M} = \mathsf{p}^{-1}[M]$ is preimage of M by p, and $f\colon M\to\mathbf{R}$ is a measurable function, then the coarea formula gives

$$\mathrm{Vol}(K)\int_M f(x)\Omega_M(dx) = \int_{\widehat{M}} f(\mathsf{p}(x))\Omega_G(d\xi).$$

In light of (5) this implies

$$\mathrm{Vol}(K)\int_M f(x)\,dx = \int_{\widehat{M}} |\mathsf{p}(\xi)|^k f(\mathsf{p}(\xi))\,\Omega_{\widehat{M}}(d\xi). \tag{6}$$

Let U,V be subspaces of of a real inner product space of dimensions k and ℓ respectively. Then the *angle*[1] between defined to be

$$\sigma(U,V) = \|u_1\wedge u_2\cdots\wedge u_k\wedge v_1\wedge v_2\wedge\cdots\wedge v_\ell\|$$

where $u_1,u_2,\dots,u_k$ is an orthonormal basis of U and $v_1,v_2,\dots,v_\ell$ is an orthonormal basis of V. More generally we if $\xi,\eta\in G$ and U is a subspace

[1] "Angle" is not really quite the correct term for $\sigma(U,V)$ as when as when U and V are one dimensional this is the sine of the usual angle between them. We use the term "angle" for brevity.

of TG_ξ and V is a subspace of TG_η, we can define the angle bwtween theses subspaces by translating back to the identity element of G:

$$\sigma(U,V)=\sigma(L_{\xi^{-1}*}U,L_{\eta^{-1}*}V)$$

where L_g is left translation by g and L_{g*} is the derivative of L_g. It is worth remarking that as the metrics $\langle\,,\rangle$ and g are conformal and elements of G are conformal maps that $\sigma(U,V)$ can be computed with respect to either metric. Let M^p and N^q be submanifolds of $\mathbf{R}^n\smallsetminus\{0\}$ and let $\widehat{M}=\pi^{-1}[M]$, $\widehat{N}=\pi^{-1}[N]$. Then by the *basic integral formula* of [3, p. 7] and using that G is unimodular for any we have for any integrable function $f\colon\widehat{M}\times\widehat{N}\to\mathbf{R}$

$$\int_G\left(\int_{\widehat{M}\cap g\widehat{N}}f(\xi,g^{-1}\xi)\Omega_{\widehat{M}\cap g\widehat{N}}(d\xi)\right)\Omega_G(dg) = \iint_{\widehat{M}\times\widehat{N}}f(\xi,\eta)\sigma(T^\perp\widehat{M}_\xi,T^\perp\widehat{N}_\eta)\,\Omega_{\widehat{M}\times\widehat{N}}(d\xi,d\eta). \tag{7}$$

We recall some definitions from [3]. If $M\subseteq\mathbf{R}^n$ then, as usual, the second fundamental form of M at $x\in M$ is a symmetric function $h\colon TM_x\times TM_x\to T^\perp M_x$. For technical reasons it is useful to consider the *extended second fundamental form* of M which is defined by letting h to a function $H\colon\mathbf{R}^n\times\mathbf{R}^n\to\mathbf{R}^n$, also symmetric and bilinear, by $H(u,v)=h(u,v)$ for $u,v\in TM_x$ and $H(u,w)=H(w,v)=0$ for $w\in T^\perp M_x$ and all $u,v\in\mathbf{R}^n$. Let

$$\text{EII}=\text{vector space of all symmetric bilinear maps from }\mathbf{R}^n\times\mathbf{R}^n\text{ to }\mathbf{R}^n.$$

This includes the extended second fundamental forms of all submanifolds of $\mathbf{R}^n$. Let

$$\operatorname{Poly}_H^\ell(\text{EII})=\begin{cases}\text{vector space of all real valued polynomials on EII}\\ \text{invariant under }H\text{ and homogeneous of degree }\ell.\end{cases}$$

If $\mathcal{P}\in\operatorname{Poly}_H^\ell(\text{EII})$ and if M is a submanifold of $\mathbf{R}^n$ set

$$I^{\mathcal{P}}(M)=\int_M\mathcal{P}(h_x^M)\,dx$$

where h^M is the extended second fundamental form of M. This is in variant under translations and the group H. That is invariant under the subdirect product $H\ltimes\mathbf{R}^n$ of maps of the form $x\mapsto ax+b$ with $a\in H$ and $b\in\mathbf{R}^n$. See [3, Section 6] for more details on these integral invariants.

Let M^p and N^q be submanifolds of $\mathbf{R}^n$ of the indicated degrees and $\mathcal{P}\in\operatorname{Poly}_H^\ell$. Our goal is to evaluate

$$\int_G I^{\mathcal{P}}(M\cap gN)\Omega_M(dg). \tag{8}$$

This is not quite the usual set up for integral as the integral invariant $I^{\mathcal{P}}$ in not invariant under the group G, but the methods of [3] carry over to this set setting.

With notation as in the Basic Integral Formula (7), let $f\colon \widehat{M}\times\widehat{N}\to\mathbf{R}$ be the function

$$f(\xi,\eta)=\mathcal{P}\left(h^{M\cap\xi\eta^{-1}N}_{\pi(\xi)}\right)|\pi(\xi)|^{p+q-n}$$

Then

$$f(\xi,g^{-1}\xi)=\mathcal{P}\left(h^{M\cap\xi\xi^{-1}gN}_{\pi(\xi)}\right)|\pi(\xi)|^{p+q-n}=\mathcal{P}\left(h^{M\cap gN}_{\pi(\xi)}\right)|\pi(\xi)|^{p+q-n}.$$

Therefore, using that $\widehat{M}\cap g\widehat{N}$ fibers over $M\cap gN$ with fiber K and that the projection is a Riemannian submersion,

$$\begin{aligned}
&\int_{\widehat{M}\cap g\widehat{N}} f(\xi,g^{-1}\xi)\,\Omega_{\widehat{M}\cap g\widehat{N}}(\xi,\eta)\\
&\qquad=\int_{\widehat{M}\cap g\widehat{N}}\mathcal{P}\left(h^{M\cap gN}_{\pi(\xi)}\right)|\pi(\xi)|^{p+q-n}\Omega_{\widehat{M}\cap g\widehat{N}}(d\xi,d\eta)\\
&\qquad=\mathrm{Vol}(K)\int_{M\cap gN}\mathcal{P}\left(h^{M\cap gN}_{x}\right)|\pi(\xi)|^{p+q-n}\Omega_{M\cap gN}(dx)\\
&\qquad=\mathrm{Vol}(K)\int_{M\cap gN}\mathcal{P}\left(h^{M\cap gN}_{x}\right)\,dx.
\end{aligned}$$

Using this in equation (7)

$$\begin{aligned}
&\mathrm{Vol}(K)\int_G\left(\int_{M\cap gN}\mathcal{P}\left(h^{M\cap gN}_{x}\right)\,dx\right)\Omega_G(dg)\\
&\qquad=\iint_{\widehat{M}\times\widehat{N}} f(\xi,\eta)\sigma(T^\perp\widehat{M}_\xi,T^\perp\widehat{N}_\eta)\,\Omega_{\widehat{M}\times\widehat{N}}(d\xi,d\eta).
\end{aligned}$$

Which is equivalent to

$$\mathrm{Vol}(K)\int_G I^{\mathcal{P}}(M\cap gN)\,\Omega_G(dg) \tag{9}$$

$$=\iint_{\widehat{M}\times\widehat{N}} f(\xi,\eta)\sigma(T^\perp\widehat{M}_\xi,T^\perp\widehat{N}_\eta)\,\Omega_{\widehat{M}\times\widehat{N}}(d\xi,d\eta). \tag{10}$$

Now to get our formula it just remains to put the right side of this equation in a usable, or at least more usable, form. Note that the map $(\xi,\eta)\mapsto(\pi(\xi),\pi(\eta))$ is a Riemannian submersion $\widehat{M}\times\widehat{N}\to M\times N$ with fibers isometric to K (where we are assuming here that M and N have the G invariant metric $\langle\,,\rangle$). Thus if for each $x\in M$ and $y\in N$ we choose $\xi_x,\eta_y\in G$ with $\pi(\xi_x)=x$ and $\pi(\eta_y)=y$ (or what is the same thing $\xi_x(\mathbf{o})=x$ and $\eta_y(\mathbf{o})=y$) we can rewrite the last equation as

$$\mathrm{Vol}(K)\int_G I^{\mathcal{P}}(M\cap gN)\,\Omega_G(g)=\iint_{M\times N}\mathcal{I}(x,y)\,\Omega_M(x)\Omega_N(y) \tag{11}$$

where

$$\begin{aligned}
\mathcal{I}(x,y) &= \iint_{K\times K} f(\xi_x a, \eta_y b)\sigma(T^\perp \widehat{M}_{\xi_x a}, T^\perp \widehat{N}_{\eta_y b})\, \Omega_K(da)\Omega_K(db) \\
&= \iint_{K\times K} \mathcal{P}\left(h_x^{M\cap \xi_x ab^{-1}\eta_y^{-1}N}\right) |x|^{p+q-n}\sigma(T^\perp \widehat{M}_{\xi_x a}, T^\perp \widehat{N}_{\eta_y b})\, \Omega_K(da)\Omega_K(db).
\end{aligned}$$

From [3, Eq. (3-13), p. 17] and that b is an isometry we have

$$\begin{aligned}
\sigma(T^\perp \widehat{M}_{\xi_x a}, T^\perp \widehat{N}_{\eta_y b}) &= \sigma((\xi_x a)_*^{-1}T^\perp M_x, (b_*^{-1}(\eta_y)_*^{-1})T^\perp N_y) \\
&= \sigma(b_*(\xi_x a)_*^{-1}T^\perp M_x, (\eta_y)_*^{-1}T^\perp N_y) \\
&= \sigma((\xi_x ab^{-1})_*^{-1}T^\perp M_x, (\eta_y)_*^{-1}T^\perp N_y).
\end{aligned}$$

Using this in equation (10)

$$\begin{aligned}
\mathcal{I}(x,y) &= \iint_{K\times K} \mathcal{P}\left(h_x^{M\cap \xi_x ab^{-1}\eta_y^{-1}N}\right) |x|^{p+q-n}\sigma(T^\perp \widehat{M}_{\xi_x a}, T^\perp \widehat{N}_{\eta_y b})\Omega_K(da)\Omega_K(db) \\
&= \iint_{K\times K} \mathcal{P}\left(h_x^{M\cap \xi_x ab^{-1}\eta_y^{-1}N}\right) |x|^{p+q-n}\sigma((\xi_x ab^{-1})_*^{-1}T^\perp M_x, (\eta_y)_*^{-1}T^\perp N_y)\Omega_K(da)\Omega_K(db) \\
&= \mathrm{Vol}(K)\int_K \mathcal{P}\left(h_x^{M\cap \xi_x a\eta_y^{-1}N}\right) |x|^{p+q-n}\sigma((\xi_x a)_*^{-1}T^\perp M_x, (\eta_y)_*^{-1}T^\perp N_y)\Omega_K(da)
\end{aligned}$$

(10)

$$= \mathrm{Vol}(K)|x|^{p+q-n}\int_K \mathcal{P}\left(h_x^{M\cap \xi_x a\eta_y^{-1}N}\right) \sigma((\xi_x a)_*^{-1}T^\perp M_x, (\eta_y)_*^{-1}T^\perp N_y)\Omega_K(da).$$

As $\mathcal{P} \in \mathrm{Poly}_H^\ell(\mathrm{EII})$, we have that $\mathcal{P}$ is homogeneous of degree ℓ. Note that if E is a $p+q-n$ dimensional submanifold of $\mathbf{R}^n$ and $\lambda > 0$ then the second fundamental form of E and its dilate λE are related by $h^{\lambda E} = (1/\lambda)h^E$. Then as $\mathcal{P}$ is homogeneous of degree ℓ we have

$$\mathcal{P}(h^{\lambda E}) = \frac{1}{\lambda^l}\mathcal{P}(h^E). \tag{12}$$

The element $\xi_x \in G$ is so that $\xi_x(\mathbf{o}) = x$ and as $|\mathbf{o}| = 1$ this implies that there is a unique element $\zeta_x \in H$ so that $\xi_x = |x|\zeta_x$. With this notation and using that $\mathcal{P}$ is invariant under the action of H,

$$\begin{aligned}
\mathcal{P}\left(h_x^{M\cap\xi_x a\eta_y^{-1}N}\right) &= \mathcal{P}\left(h_x^{\xi_x(\xi_x^{-1}M\cap a\eta_y^{-1}N)}\right) \\
&= \mathcal{P}\left(h_x^{|x|\zeta_x(\xi_x^{-1}M\cap a\eta_y^{-1}N)}\right) \\
&= \frac{1}{|x|^\ell}\mathcal{P}\left(h_{\mathbf{o}}^{\zeta_x(\xi_x^{-1}M\cap a\eta_y^{-1}N)}\right) \\
&= \frac{1}{|x|^\ell}\mathcal{P}\left(h_{\mathbf{o}}^{(\xi_x^{-1}M\cap a\eta_y^{-1}N)}\right) \\
&= \frac{1}{|x|^\ell}\mathcal{P}\left(h_{\mathbf{o}}^{a^{-1}\xi_x^{-1}M\cap\eta_y^{-1}N}\right) \\
&= \frac{1}{|x|^\ell}\mathcal{P}\left(h_{\mathbf{o}}^{(\xi_x a)^{-1}M\cap\eta_y^{-1}N}\right).
\end{aligned}$$

Putting this in (11)

$$\begin{aligned}
&\mathcal{I}(x,y) \\
&= \mathrm{Vol}(K)|x|^{p+q-n-\ell}\int_K \mathcal{P}\left(h_{\mathbf{o}}^{(\xi_x a)^{-1}M\cap\eta_y^{-1}N}\right)\sigma((\xi_x a)_*^{-1}T^\perp M_x,(\eta_y)_*^{-1}T^\perp N_y)\Omega_k(da) \\
&= \mathrm{Vol}(K)|x|^{p+q-n-\ell}I_K^{\mathcal{P}}(T(\xi_x^{-1}M)_{\mathbf{o}},h_{\mathbf{o}}^{\xi_x^{-1}M},T(\eta_y^{-1}N)_{\mathbf{o}},h_{\mathbf{o}}^{\eta_y^{-1}N})
\end{aligned}$$

where $I_K^{\mathcal{P}}(V,h_1,W,h_2)$ is as defined in [3, Def 6.3 p. 40]. Putting this into equation (11) gives

$$\begin{aligned}
&\int_G I^{\mathcal{P}}(M\cap gN)\,\Omega_G(g) \\
&\quad= \iint_{M\times N}|x|^{p+q-n-\ell}I_K^{\mathcal{P}}(T(\xi_x^{-1}M)_{\mathbf{o}},h_{\mathbf{o}}^{\xi_x^{-1}M},T(\eta_y^{-1}N)_{\mathbf{o}},h_{\mathbf{o}}^{\eta_y^{-1}N})\,\Omega_M(dx)\Omega_N(dy) \\
&\quad= \iint_{M\times N}|x|^{q-n-\ell}|y|^{-q}I_K^{\mathcal{P}}(T(\xi_x^{-1}M)_{\mathbf{o}},h_{\mathbf{o}}^{\xi_x^{-1}M},T(\eta_y^{-1}N)_{\mathbf{o}},h_{\mathbf{o}}^{\eta_y^{-1}N})\,dxdy.
\end{aligned} \tag{14}$$

To simplify this a little more it would be nice to get rid of the dependence on the elements ξ_x and η_y. This can be done by using the dilation property $h^{\lambda M}=(1/\lambda)h$, and that $\xi_x=|x|\zeta_x$ for some $\zeta_x\in H$. Thus

$$h_{\mathbf{o}}^{\xi_x M}=h_{\mathbf{o}}^{|x|^{-1}\zeta_x^{-1}M}=|x|h_{|x|\mathbf{o}}^{\zeta_x^{-1}M}$$

which when used along with the invariance properties of $I_K^{\mathcal{P}}$ in equation (14) gives

Proposition 3. *With notation as above the kinematic formula*

$$\begin{aligned}\int_G & I^{\mathcal{P}}(M\cap gN)\,\Omega_G(dg)\\ &= \iint_{M\times N} |x|^{q-n-\ell}|y|^{-q} I_K^{\mathcal{P}}(TM_x, |x|h_x^M, TN_y, |y|h_y^N)\,dx\,dy\\ &= \iint_{M\times N} |x|^{p+q-n-\ell} I_K^{\mathcal{P}}(TM_x, |x|h_x^M, TN_y, |y|h_y^N)\Omega_M(dx)\Omega_N(dy).\end{aligned}$$

holds. □

Unfortunately there does seem to be any obvious way to farther simplify the integrand $I_K^{\mathcal{P}}(TM_x, |x|h_x^M, TN_y, |y|h_y^N)$.

With $\mathrm{Gr}(L_0, G)$ as in (2) and letting

$$J_K^{\mathcal{P}}(TM_x, h_x^M) = I_K^{\mathcal{P}}(TM_x, |x|h_x^M, L_0, 0) = |x|^\ell I_K^{\mathcal{P}}(TM_x, h_x^M, L_0, 0)$$

we can combine Proposition 3 and Equations (4) and (6) to get:

Corollary 4. *If $\mathcal{P}\in \mathrm{Poly}_H^\ell$ and L_0 has $\dim(L_0)=q$, then for any compact submanifold M of $\mathbf{R}^n\setminus\{0\}$ with $\dim(M)=p$*

$$\begin{aligned}\int_{\mathrm{Gr}(L_0,G)} I^{\mathcal{P}}(M\cap L)\,dL &= C\int_M |x|^{p+q-n} J^{\mathcal{P}}(TM_x, h_x^M)\Omega_M(dx)\\ &= C\int_M |x|^{q-n} J^{\mathcal{P}}(TM_x, h_x^M)\,dx.\end{aligned}$$

where C is a constant that only depends on the choice of measure on $\mathrm{Gr}(L_0,G)$ (or equivalently only on the total volume of $\mathrm{Gr}(L_0,G)$.) □

3. Poincare and Crofton type formulas.

We need a bit of notation. Let $V\subseteq T(\mathbf{R}^n)_x$ and $W\subseteq T(\mathbf{R}^n)_y$ where $x,y\neq 0$. Choose $\xi_x,\eta_y\in G$ so that $\xi_x(x)=\mathbf{o}$ and $\eta_y(y)=\mathbf{o}$ and define the *averaged angle* between V and W to be

$$\sigma_K(V,W) := \int_K \sigma(\xi_{x*}^{-1}V, a_*\eta_{y*}^{-1}W)\Omega_K(da).$$

That is use elements ξ_x and η_y move the subspaces V and W to the point $\mathbf{o}$ and then average the angle between the resulting subspace with respect to all possible choices of η_y. This satisfies, cf. [3, Prop. 3.4 Page 14]

$$\sigma_K(V,W) = \sigma_K(W,V) = \sigma_K(g_*V,W) = \sigma_K(V,g_*W) \tag{13}$$

for all $g\in G$.

Then $\ell=0$ and the formula for $I_K^{\mathcal{P}}(TM_x, |x|h_x^M, TN_y, |y|h_y^N)$ simplifies to

$$I_K^{\mathcal{P}}(TM_x, |x|h_x^M, TN_y, |y|h_y^N) = \sigma_K(TM_x^\perp, TN_y^\perp).$$

This follows from the formula for I_K [3, Page 40] upon letting $\mathcal{P}\equiv 1$. So in this case the kinematic formula becomes

Proposition 5. *If M^p and N^q are compact submanifolds of $\mathbf{R}^n \smallsetminus \{0\}$*

$$\begin{aligned}\int_G \operatorname{Vol}(M\cap N)\Omega_G(dg) &= \iint_{M\times N} |x|^{q-n}|y|^{-q}\sigma_K(T^\perp M_x, T^\perp N_y)\,dx\,dy\\ &= \iint_{M\times N} |x|^{p+q-n}\sigma_K(T^\perp M_x, T^\perp N_y)\Omega_M(dx)\Omega_N(dy).\end{aligned}$$

And Corollary 4 becomes

Corollary 6. *With the hypothesis of Corollary 4*

$$\begin{aligned}\int_{\operatorname{Gr}(L_0,G)} \operatorname{Vol}(M\cap L)\,dL &= C\int_M |x|^{p+q-n}\sigma_K(T^\perp M_x, L_0^\perp)\Omega_M(dx)\\ &= C\int_M |x|^{q-n}\sigma_K(T^\perp M_x, L_0^\perp)\,dx.\end{aligned}$$

with the constant C only depending on the total volume of $\operatorname{Gr}(L_0,G)$. □

References

[1] Armand Borel, *Le plan projectif des octaves et les sphères comme espaces homogènes*, C. R. Acad. Sci. Paris **230** (1950), 1378–1380. MR 34768

[2] Phillip A. Griffiths, *Complex differential and integral geometry and curvature integrals associated to singularities of complex analytic varieties*, Duke Math. J. **45** (1978), no. 3, 427–512. MR 507455

[3] Ralph Howard, *The kinematic formula in Riemannian homogeneous spaces*, Mem. Amer. Math. Soc. **106** (1993), no. 509, vi+69. MR 1169230

[4] Deane Montgomery and Hans Samelson, *Transformation groups of spheres*, Ann. of Math. **44** (1943), no. 3, 454–470. MR 8817

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