CONTINUOUS NOWHERE DIFFERENTIABLE MULTIVARIATE FUNCTIONS

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ABSTRACT. Let U be an open set in $\mathbb{R}^d$. A continuous function $f: U \to \mathbb{R}$ is ***strongly nowhere differentiable*** if and only if for each $\gamma \in (0,1]$ and for each unit speed $C^{1,\gamma}$ curve $c: [a,b] \to U$, the composition $f \circ c: [a,b] \to \mathbb{R}$ is nowhere differentiable on (a,b). For bounded U, let $\overline{U}$ be the closure of U and $C(\overline{U})$ be the Banach space of continuous real-valued functions on $\overline{U}$ with the sup norm.
Theorem. *In the sense of the Baire category theorem, almost every $f \in C(\overline{U})$ is strongly nowhere differentiable on U.*

1. Introduction

The existence of continuous nowhere differentiable functions of a single variable has been known since the 19th century. The earliest proposed example is due to Riemann who claimed the function $R(x) = \sum_{n=1}^{\infty} n^{-2} \sin(\pi n^2 x)$ was nowhere differentiable. In 1916, G. H. Hardy proved the function R was non-differentiable at all irrational points and all rational points of the form $(2p+1)/(2q)$ or $2p/(4q+1)$ with p and q integers [6]. In 1970, Gerver [5] proved $R'(x) = -\pi/2$ for rational points of the form $x = (2p+1)/(2q+1)$. So Riemann's example is only almost everywhere non-differentiable. In 1872, Weierstrass introduced his famous example $W(x) = \sum_{n=1}^{\infty} a^n \cos(b^n \pi x)$ and proved the function W is nowhere differentiable when b is an odd integer, $a \in (0,1)$, and $ab > 1 + 3\pi/2$.

By the 1930s it was realized that not only do continuous nowhere differentiable functions exist, but also they are in some sense typical: Banach [1] and Mazurkiewicz [9] showed that in $C([0,1])$, the metric space of continuous real valued functions with the supremum norm, the set of nowhere differentiable functions contains a dense G_δ set.

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Therefore, in the sense of the Baire category theorem, almost every continuous function is nowhere differentiable.

For a nice discussion of Baire's theorem and its application to ideas related to nowhere differentiable functions see the beautiful monograph of Oxtoby [10]. Jarnicki and Plug's comprehensive book [7] on nowhere differentiable functions has ample constructions along with history.

In this note we consider analogous questions for multivariate functions. Let $U \subseteq \mathbb{R}^d$ be open and $f\colon U \to \mathbb{R}$. The standard definition of f being differentiable at $x_0 \in U$ is that there is a linear map $f'(x_0)\colon \mathbb{R}^d \to \mathbb{R}$ such that

$$\|f(x) - f(x_0) - f'(x_0)(x - x_0)\| = \|x - x_0\|\epsilon(x; x_0)$$

where $\lim_{x\to x_0} \epsilon(x; x_0) = 0$. Let $g_j\colon \mathbb{R} \to \mathbb{R}$ be a continuous nowhere differentiable function of one variable. Then the function

$$f(x_1, x_2, \dots, x_d) = g_1(x_1) + g_2(x_2) + \cdots + g_d(x_d)$$

is nowhere differentiable; yet, its restriction to a lower dimensional submanifold can be everywhere differentiable. For example if $d = 2$ and $g_2 = -g_1$ then $f(x, y) = g_1(x) - g_1(y)$ is nowhere differentiable on the plane, but $f(x, x) \equiv 0$ and so f restricted to the line $y = x$ is C^∞.

Recall $g : I \to \mathbb{R}^d$, where I is an interval in $\mathbb{R}$, is α-Hölder continuous (in short, is $C^{0,\alpha}$) where $\alpha > 0$ if and only if there is a constant ρ so that $\|g(x_2) - g(x_1)\| \leq \rho|x_2 - x_1|^\alpha$ for all $x_1, x_2 \in I$. Also, $g \in C^{1,\alpha}$ if and only if its first derivative exists and $g' \in C^{0,\alpha}$.

Definition 1.1. A ***test curve*** in $\mathbb{R}^d$ is a C^1 function $c\colon [a, b] \to \mathbb{R}^d$, defined on an interval $[a, b]$, satisfying c has unit speed (i.e., $\|c'(s)\| = 1$ for each $s \in [a, b]$) and c is $C^{1,\gamma}$ for some $\gamma \in (0, 1]$ (i.e., there is a constant ρ so that $\|c'(s_2) - c'(s_1)\| \leq \rho|s_2 - s_1|^\gamma$ for all $s_1, s_2 \in [a, b]$).

Definition 1.2. Let $f\colon U \to \mathbb{R}$ where U is an open set in $\mathbb{R}^d$. Then f is ***strongly nowhere differentiable*** if and only if for every test curve $c\colon [a, b] \to U$ the composition $f \circ c$ is nowhere differentiable on (a, b).

Let f be strongly nowhere differentiable on U and $x \in U$. Then f has no directional derivative at x. Indeed, let $v \in \mathbb{R}^d$ with $\|v\| = 1$. The line segment parameterized by $c(s) = x + sv$ for $s \in [-\delta, \delta]$ with δ small is a U-valued test curve so $f \circ c$ has no derivative at 0.

For a bounded set U in $\mathbb{R}^d$ with closure $\overline{U}$, let $C(\overline{U})$ be the Banach space of all continuous functions $f\colon \overline{U} \to \mathbb{R}$ with the supremum norm.

Main Theorem. *Let U be a bounded open set in $\mathbb{R}^d$ with closure $\overline{U}$ and let $C(\overline{U})$ be the Banach space of continuous functions $f\colon \overline{U} \to \mathbb{R}$ endowed with the supremum norm. The set of functions in $C(\overline{U})$ that* *are strongly nowhere differentiable on U contains a G_δ subset that is dense in $C(\overline{U})$.*

More informally, in the sense of the Baire Category Theorem, for almost every $f \in C(\overline{U})$ the restriction $f\big|_U$ is strongly nowhere differentiable. The Main Theorem is proved in Section 3.

2. Construction of auxiliary functions

Definition 2.1. A function $f\colon \mathbb{R} \to \mathbb{R}$ is ***precisely*** $\boldsymbol{C^{0,\alpha}}$ where $\alpha > 0$ if and only if there is a constant $C > 0$ so that $|f(x)-f(y)| \le C|x-y|^\alpha$ for all $x, y \in \mathbb{R}$, and for all $\beta > \alpha$ and $y \in \mathbb{R}$

$$\limsup_{x \to y} \frac{|f(x) - f(y)|}{|x-y|^\beta} = \infty.$$

Definition 2.2. Let I be an interval of $\mathbb{R}$ and $x_0 \in I$. A function $f\colon I \to \mathbb{R}$ is ***precisely*** $\boldsymbol{C^{0,\alpha}}$ ***at*** $\boldsymbol{x_0}$ where $\alpha > 0$ if and only if

$$\limsup_{x \to x_0} \frac{|f(x) - f(x_0)|}{|x-x_0|^\alpha} < \infty$$

and for any $\beta > \alpha$

$$\limsup_{x \to x_0} \frac{|f(x) - f(x_0)|}{|x-x_0|^\beta} = \infty. \tag{2.1}$$

Theorem 2.3. *For each α with $0 < \alpha < 1$ there is a precisely $C^{0,\alpha}$ function $f\colon \mathbb{R} \to \mathbb{R}$, which is also even, 2-periodic, and thus bounded.*

Proof. See the appendix. □

Proposition 2.4. *If $f\colon [a,b] \to \mathbb{R}$ is precisely $C^{0,\alpha}$ at $x_0 \in [a,b]$, with $0 < \alpha < 1$, then f is not differentiable at x_0.*

Proof. This is a direct consequence of the definition of precisely $C^{0,\alpha}$ at x_0. As $\alpha < 1$, taking $\beta = 1$ gives

$$\limsup_{x \to x_0} \left| \frac{f(x) - f(x_0)}{x - x_0} \right| = \limsup_{x \to x_0} \frac{|f(x) - f(x_0)|}{|x-x_0|^1} = \infty.$$

Thus f is not differentiable at x_0. □

Lemma 2.5. *If $\alpha_1, \alpha_2, \dots, \alpha_n \in (0,1]$ are distinct and f_j is precisely C^{0,α_j} at x_0, then the sum $f = f_1 + f_2 + \dots + f_n$ is precisely $C^{0,\alpha}$ at x_0 where $\alpha = \min_{1 \le j \le n} \alpha_j$.*

Proof. By induction it is enough to show Lemma 2.5 for $n = 2$. Without loss of generality, assume $\alpha_1 < \alpha_2$. Then

$$\frac{|f(x) - f(x_0)|}{|x - x_0|^{\alpha_1}} \le \frac{|f_1(x) - f_1(x_0)|}{|x - x_0|^{\alpha_1}} + \frac{|f_2(x) - f_2(x_0)|}{|x - x_0|^{\alpha_2}}|x - x_0|^{\alpha_2 - \alpha_1}. \tag{2.2}$$

The function f_2 is precisely C^{0,α_2} at x_0 and therefore

$$\limsup_{x\to x_0} \frac{|f_2(x) - f_2(x_0)|}{|x - x_0|^{\alpha_2}} < \infty.$$

Also $\lim_{x\to x_0} |x - x_0|^{\alpha_2 - \alpha_1} = 0$ as $\alpha_1 < \alpha_2$. Thus

$$\lim_{x\to x_0} \frac{|f_2(x) - f_2(x_0)|}{|x - x_0|^{\alpha_2}}|x - x_0|^{\alpha_2 - \alpha_1} = 0. \tag{2.3}$$

Using this and that f_1 is precisely C^{0,α_1} at x_0, inequality (2.2) gives

$$\limsup_{x\to x_0} \frac{|f(x) - f(x_0)|}{|x - x_0|^{\alpha_1}} \le \limsup_{x\to x_0} \frac{|f_1(x) - f_1(x_0)|}{|x - x_0|^{\alpha_1}} + 0 < \infty.$$

Let $\beta \in (\alpha_1, \alpha_2)$. By the reverse triangle inequality

$$\frac{|f(x) - f(x_0)|}{|x - x_0|^{\beta}} \ge \frac{|f_1(x) - f_1(x_0)|}{|x - x_0|^{\beta}} - \frac{|f_2(x) - f_2(x_0)|}{|x - x_0|^{\alpha_2}}|x - x_0|^{\alpha_2 - \beta}. \tag{2.4}$$

Since $\beta < \alpha_2$, the argument showing (2.3) holds gives

$$\lim_{x\to x_0} \frac{|f_2(x) - f_2(x_0)|}{|x - x_0|^{\alpha_2}}|x - x_0|^{\alpha_2 - \beta} = 0.$$

Thus since $\alpha_1 < \beta$ and f_1 is precisely C^{0,α_1} at x_0, by (2.4)

$$\limsup_{x\to x_0} \frac{|f(x) - f(x_0)|}{|x - x_0|^{\beta}} \ge \limsup_{x\to x_0} \frac{|f_1(x) - f_1(x_0)|}{|x - x_0|^{\beta}} - 0 = \infty.$$

Note for any $\varepsilon > 0$

$$\limsup_{x\to x_0} \frac{|f(x) - f(x_0)|}{|x - x_0|^{\beta+\varepsilon}} \ge \limsup_{x\to x_0} \frac{|f(x) - f(x_0)|}{|x - x_0|^{\beta}} = \infty.$$

Thus f is precisely C^{0,α_1} at x_0. □

Lemma 2.6. *Let $u \colon J \to I$ be a C^1 function for intervals I and J of $\mathbb{R}$. Let $f \colon I \to \mathbb{R}$ is precisely $C^{0,\alpha}$ at $u(x_0)$ where x_0 is in the interior of J.*

(a) If $u'(x_0) \neq 0$ then $f \circ u$ is precisely $C^{0,\alpha}$ at x_0.

Furthermore let the curve u be $C^{1,\gamma}$ for some $\gamma \in (0, 1]$ and $\alpha > \frac{1}{1+\gamma}$.

(b) If $u'(x_0) = 0$ then $f \circ u$ is differentiable at x_0 and $(f \circ u)'(x_0) = 0$.

Proof. For (a) let $u'(x_0) \neq 0$. Then, as u' is continuous, there is an interval $J_0 = (x_0 - \delta, x_0 + \delta)$ such that u' does not change sign on J_0 and, for some constant $C > 0$, we have $C^{-1} \leq |u'(x)| \leq C$ for each $x \in J_0$. So by the mean value theorem

$$\frac{1}{C}|x - \widetilde{x}| \leq |u(x) - u(\widetilde{x})| \leq C|x - \widetilde{x}|$$

holds for $x, \widetilde{x} \in J_0$. Let $I_0 = \{u(x) : x \in J_0\}$. The last inequality gives that u is a Lipschitz bijection from J_0 onto I_0 with a Lipschitz inverse. Hence, since f is precisely $C^{0,\alpha}$ at $u(x_0)$,

$$\begin{aligned}
&\limsup_{x\to x_0} \frac{|f(u(x)) - f(u(x_0))|}{|x - x_0|^\alpha} \\
&\qquad = \limsup_{x\to x_0} \frac{|f(u(x)) - f(u(x_0))|}{|u(x) - u(x_0)|^\alpha} \left|\frac{u(x) - u(x_0)}{x - x_0}\right|^\alpha \\
&\qquad \leq C^\alpha \limsup_{x\to x_0} \frac{|f(u(x)) - f(u(x_0))|}{|u(x) - u(x_0)|^\alpha} \\
&\qquad = C^\alpha \limsup_{y\to u(x_0)} \frac{|f(y) - f(u(x_0))|}{|y - u(x_0)|^\alpha} \\
&\qquad < \infty
\end{aligned}$$

where we have done the change of variable $y = u(x)$ in the last $\limsup$.

Similarly, for each $\beta > \alpha$

$$\begin{aligned}
&\limsup_{x\to x_0} \frac{|f(u(x)) - f(u(x_0))|}{|x - x_0|^\beta} \\
&\qquad = \limsup_{x\to x_0} \frac{|f(u(x)) - f(u(x_0))|}{|u(x) - u(x_0)|^\beta} \left|\frac{u(x) - u(x_0)}{x - x_0}\right|^\beta \\
&\qquad \geq \frac{1}{C^\beta} \limsup_{x\to x_0} \frac{|f(u(x)) - f(u(x_0))|}{|u(x) - u(x_0)|^\beta} \\
&\qquad = \frac{1}{C^\beta} \limsup_{y\to u(x_0)} \frac{|f(y) - f(u(x_0))|}{|y - u(x_0)|^\beta} \\
&\qquad = \infty.
\end{aligned}$$

Thus $f \circ u$ is precisely $C^{0,\alpha}$ at x_0.

For (b) we have $u'(x_0) = 0$ and constants $\gamma \in (0, 1]$ and $\rho > 0$ so that $|u'(x) - u'(y)| \leq \rho|x - y|^\gamma$ for $x, y \in J$. By the mean value theorem, for x in the interior of J there is a ξ between x and x_0 with

$$\begin{aligned}
|u(x) - u(x_0)| &= |u'(\xi)||x - x_0| \\
&= |u'(\xi) - u'(x_0)||x - x_0| \quad (\text{as } u'(x_0) = 0)
\end{aligned}$$

$$\begin{aligned} &\leq \rho|\xi - x_0|^\gamma \, |x - x_0| \\ &\leq \rho|x - x_0|^\gamma \, |x - x_0| \qquad\qquad \text{(as } \xi \text{ between } x \text{ and } x_0\text{)} \\ &= \rho|x - x_0|^{1+\gamma}. \end{aligned}$$

Since f is precisely $C^{0,\alpha}$ at $u(x_0)$, there is a constant $K > 0$ such that $|f(y) - f(u(x_0))| \leq K|y - u(x_0)|^\alpha$ for y in some neighborhood of $u(x_0)$. Thus for x close to x_0

$$|f(u(x)) - f(u(x_0))| \leq K|u(x) - u(x_0)|^\alpha \leq K\rho^\alpha |x - x_0|^{\alpha(1+\gamma)}$$

and $\alpha(1 + \gamma) > 1$ by the lower bound on α thus, $(f \circ u)'(x_0) = 0$. □

Theorem 2.7. *Let $\gamma \in (0, 1]$ and $\alpha_1, \alpha_2, \ldots, \alpha_d$ be distinct numbers in the interval $(\frac{1}{1+\gamma}, 1)$. For each $j \in \{1, 2, \ldots, d\}$, let the function $f_j \colon \mathbb{R} \to \mathbb{R}$ be precisely C^{0,α_j} at each point in $\mathbb{R}$. Define $f \colon \mathbb{R}^d \to \mathbb{R}$ by*

$$f(x_1, x_2, \ldots, x_d) = \sum_{j=1}^{d} f_j(x_j).$$

Then f is continuous and, for any $C^{1,\gamma}$ test curve $c : [a, b] \to \mathbb{R}^d$, the composite $f \circ c$ is nowhere differentiable on (a, b).

Moreover, for any $C^{1,\gamma}$ test curve $c \colon [a, b] \to \mathbb{R}^d$ and any $s_0 \in (a, b)$, there is an $\alpha \in \{\alpha_1, \alpha_2, \ldots, \alpha_d\}$ such that $f \circ c$ is precisely $C^{0,\alpha}$ at s_0.

A concrete choice of the function f_j is to let f_j be the precisely $C^{0,\alpha}$ function constructed in Theorem 2.3 with $\alpha = \alpha_j$.

Proof. Note f is continuous since it is the sum of continuous function. Let $\gamma \in (0, 1]$. Let $c \colon [a, b] \to \mathbb{R}^d$ be a $C^{1,\gamma}$ test curve. Fix $s_0 \in (a, b)$. By Proposition 2.4, it only remains to show $f \circ c$ is precisely C^{0,α_j} at s_0 for some j.

Write the test curve c as

$$c(s) = (u_1(s), u_2(s), \ldots, u_d(s)).$$

Then each function u_j is C^1. In this notation

$$(f \circ c)(s) = \sum_{j=1}^{d} (f_j \circ u_j)(s).$$

As $c \in C^{1,\gamma}$, there is a $\rho > 0$ so that $\|c'(s_2) - c'(s_1)\| \leq \rho|s_2 - s_1|^\gamma$; thus, $|u_j'(s_2) - u_j'(s_1)| \leq \rho|s_2 - s_1|^\gamma$ for all j and $s_1, s_2 \in [a, b]$. Decompose $f \circ c$ as $f \circ c = f_A + f_B$ where

$$A = \{j : u_j'(s_0) \neq 0\}, \qquad B = \{j : u_j'(s_0) = 0\}$$

and, following the convention that the sum over the empty set is zero,

$$f_A = \sum_{j\in A} f_j \circ u_j, \qquad f_B = \sum_{j\in B} f_j \circ u_j.$$

By Lemma 2.6, $f_j \circ u_j$ is precisely C^{0,α_j} at s_0 when $j \in A$ while $f_j \circ u_j$ has $(f_j \circ u_j)'(s_0) = 0$ when $j \in B$. As $c'(s_0) \neq 0$, it follows $A \neq \varnothing$; let $\alpha = \min_{j\in A} \alpha_j$. By Lemma 2.5, f_A precisely $C^{0,\alpha}$ at s_0. As f_B is differentiable at s_0, the sum $f \circ c = f_A + f_B$ is precisely $C^{0,\alpha}$ at s_0. $\square$

3. Proof of the main theorem

Let U be a bounded open set in $\mathbb{R}^d$. Let $\overline{U}$ be the closure of U and ∂U be the boundary of U. For a nonnegative integer n, let

$$K_n := \{u \in U : \|u - x\| \geq 1/n \text{ for each } x \in \partial U\}$$

be the set of points in U that are a distance of at least $1/n$ from ∂U.

Definition 3.1. Let n be a positive integer and $\gamma \in (0, 1]$. Let $\mathcal{C}^\gamma_n(U)$ be the set of test curves $c\colon [-1/n, 1/n] \to U$ satisfying

(a) $\|c'(s_2) - c'(s_1)\| \leq n|s_2 - s_1|^\gamma$ for all $s_1, s_2 \in [-1/n, 1/n]$

(b) $\|c(s) - x\| \geq 1/n$ for all $s \in [-1/n, 1/n]$ and $x \in \partial U$.

Lemma 3.2. *The set $\mathcal{C}^\gamma_n(U)$ is compact in the C^1 topology.*

Recall the C^1 topology on C^1 functions $c\colon [a, b] \to U$ is given by the norm $\|c\|_{C^1} = \|c\|_{L^\infty} + \|c'\|_{L^\infty}$.

Proof. Note K_n is a closed bounded, and thus compact, subset of $\mathbb{R}^d$. Condition (b) in Definition 3.1 can be restated as $c\colon [-1/n, 1/n] \to K_n$. Let $\langle c_\ell\rangle_{\ell=1}^\infty$ be a sequence from $\mathcal{C}^\gamma_n(U)$. It suffices to find a subsequence of $\langle c_\ell\rangle_{\ell=1}^\infty$ that converges in the C^1 topology to an element of $\mathcal{C}^\gamma_n(U)$.

The set $\{c'\colon c \in \mathcal{C}^\gamma_n(U)\}$ is uniformly equicontinuous as the functions c' satisfy the uniform Hölder condition in (a) and this set is bounded in the sup norm topology as the curves are unit speed. Therefore the Arzelà–Ascoli Theorem (cf. [4, p. 137]) implies the closure of $\{c'\colon c \in \mathcal{C}^\gamma_n(U)\}$ is compact sup norm topology. So we can pass to a subsequence and assume $\langle c'_\ell\rangle_{\ell=1}^\infty$ converges uniformly on $[-1/n, 1/n]$ to some function. By the compactness of K_n we can pass to a further subsequence and assume $\langle c_\ell(0)\rangle_{\ell=1}^\infty$ converges to some point in K_n. Thus (cf. [2, §8.2]) there is a differentiable function $c\colon [-1/n, 1/n] \to \mathbb{R}^d$ such that

$$\lim_{\ell\to\infty} c_\ell = c$$

$$\lim_{\ell\to\infty} c'_\ell = c'$$

with both convergences being uniformly on $[-1/n, 1/n]$. So c has unit speed as each c_ℓ does. Condition (a) holds since if $s_1, s_2 \in [-1/n, 1/n]$ then $\|c'(s_2) - c'(s_1)\| = \lim_{\ell\to\infty} \|c'_\ell(s_2) - c'_\ell(s_1)\| \le n|s_2 - s_1|^\gamma$. The curve c is K_n-valued as each c_ℓ is and K_n is closed. Thus $c \in \mathcal{C}_n^\gamma(U)$. $\square$

Definition 3.3. Let n be a positive integer and $\gamma \in (0, 1]$. Let $\mathcal{F}_n^\gamma(U)$ be the set of $f \in C(\overline{U})$ such that for some $c \in \mathcal{C}_n^\gamma(U)$ the inequality

$$|f(c(s)) - f(c(0))| \le n|s| \tag{3.1}$$

holds for all $s \in [-1/n, 1/n]$.

Lemma 3.4. *The set $\mathcal{F}_n^\gamma(U)$ is a closed nowhere dense subset of $C(\overline{U})$.*

Proof. Let $\gamma \in (0, 1]$. Let $\langle f_\ell\rangle_{\ell=1}^\infty$ be a sequence in $\mathcal{F}_n^\gamma(U)$ with $f_\ell \to f$ uniformly on $\overline{U}$ for some $f \in C(\overline{U})$. To show $\mathcal{F}_n^\gamma(U)$ is closed it is enough to show $f \in \mathcal{F}_n^\gamma(U)$. For each ℓ there is $c_\ell \in \mathcal{C}_n^\gamma(U)$ with

$$|f_\ell(c_\ell(s)) - f_\ell(c_\ell(0))| \le n|s|$$

for all $s \in [-1/n, 1/n]$. Since $\mathcal{C}_n^\gamma(U)$ is compact, passing to a subsequence, we assume there is a $c \in \mathcal{C}_n^\gamma(U)$ such that $c_\ell \to c$ in the C^1 topology. Then

$$\lim_{\ell\to\infty} f_\ell(c_\ell(s)) = f(c(s))$$

for each $s \in [-1/n, 1/n]$ since f is continuous, $c_\ell(s) \to c(s)$, and

$$\begin{aligned}|f_\ell(c_\ell(s)) - f(c(s))| &\le |f_\ell(c_\ell(s)) - f(c_\ell(s))| + |f(c_\ell(s)) - f(c(s))|\\ &\le \|f_\ell - f\|_{L^\infty} + |f(c_\ell(s)) - f(c(s))|\\ &\xrightarrow{\ell\to\infty} 0.\end{aligned}$$

Therefore

$$|f(c(s)) - f(c(0))| = \lim_{\ell\to\infty} |f_\ell(c_\ell(s)) - f_\ell(c_\ell(0))| \le n|s|$$

for all $s \in [-1/n, 1/n]$. Thus $f \in \mathcal{F}_n^\gamma(U)$. So $\mathcal{F}_n^\gamma(U)$ is closed.

It remains to show $\mathcal{F}_n^\gamma(U)$ is nowhere dense. Toward a contradiction assume this is not the case. Then the closed set $\mathcal{F}_n^\gamma(U)$ has nonempty interior. Thus $\mathcal{F}_n^\gamma(U)$ contains a nonempty open subset $\mathcal{V}$ of $C(\overline{U})$. As the smooth functions are dense in $C(\overline{U})$ there is a $f_0 \in \mathcal{V}$ that is C^∞. Let f be the function constructed in Theorem 2.7.

For a small enough positive δ the sum $f_\delta := f_0 + \delta f$ is in the open set $\mathcal{V}$ and so $f_\delta \in \mathcal{F}_n^\gamma(U)$. By the definition of $\mathcal{F}_n^\gamma(U)$ there is $c \in \mathcal{C}_n^\gamma(U)$ so that the Lipschitz inequality

$$|f_\delta(c(s)) - f_\delta(c(0))| \le n|s|$$

holds for each $s \in [-1/n, 1/n]$. However, by Theorem 2.7, $f \circ c$ is precisely $C^{0,\alpha}$ at 0 for some $\alpha \in (0, 1)$ since the curve c is $C^{1,\gamma}$. As f_0

is smooth, $f_0 \circ c$ is C^1. Thus the sum $f_\delta \circ c = f_0 \circ c + \delta(f \circ c)$ is precisely $C^{0,\alpha}$ at 0 where $\alpha < 1$. Hence

$$\limsup_{s \to 0} \frac{|f_\delta(c(s)) - f_\delta(c((0))|}{|s - 0|} = \infty,$$

contradicting the previous inequality. So $\mathcal{F}_n^\gamma(U)$ is nowhere dense. $\square$

Proof of the Main Theorem. Let

$$\mathcal{G} := C(\overline{U}) \setminus \left(\bigcup_{n=1}^{\infty} \mathcal{F}_n^{1/n}(U) \right).$$

Then $\mathcal{G}$ is a G_δ set that is dense in $C(\overline{U})$ by the Baire category theorem and Lemma 3.4. It suffices to show the functions in $\mathcal{G}$ are strongly nowhere differentiable on U. Let $f \in \mathcal{G}$.

Let $c\colon [a, b] \to U$ be a test curve. Towards a contradiction, assume $f \circ c$ is differentiable at some $s_0 \in (a, b)$. Then there is a constant M such that for each $s \in [a, b]$

$$|(f \circ c)(s) - (f \circ c)(s_0)| \leq M|s - s_0|. \tag{3.2}$$

As c is a test curve, there is a $\gamma \in (0, 1]$ and a constant ρ so that $\|c'(s_2) - c'(s_1)\| \leq \rho|s_2 - s_1|^\gamma$ for $s_1, s_2 \in [a, b]$. Choose n so that $n \geq \max(M, \frac{1}{\gamma}, \rho, 2)$ and $[s_0 - 1/n, s_0 + 1/n] \subseteq (a, b)$. The image of c, being compact, is a positive distance from the boundary of the open set U; thus, by possibly increasing the size of n, we can also assume $c(s) \in K_n$ for each $s \in [a, b]$.

Define $\tilde{c}\colon [-1/n, 1/n] \to U$ by

$$\tilde{c}(t) = c(t + s_0).$$

Then $\tilde{c}$ is of unit speed and K_n-valued as c is. For $t_1, t_2 \in [-1/n, 1/n]$, we have $|t_2 - t_1| \leq 1$ as $n \geq 2$ and so, since $\rho \leq n$ and $1/n \leq \gamma$,

$$\begin{aligned} \|\tilde{c}'(t_2) - \tilde{c}'(t_1)\| &= \|c'(t_2 + s_0) - c'(t_1 + s_0)\| \\ &\leq \rho|t_2 - t_1|^\gamma \leq n|t_2 - t_1|^{1/n}. \end{aligned}$$

Thus $\tilde{c} \in \mathcal{C}_n^{1/n}(U)$. The inequality in (3.2) along with $M \leq n$ give, for each $t \in [-1/n, 1/n]$,

$$|f(\tilde{c}(t)) - f(\tilde{c}(0))| = |f(c(t + s_0)) - f(c(s_0))| \leq M|t| \leq n|t|.$$

Hence $f \in \mathcal{F}_n^{1/n}(U)$, which contradicts $f \in \mathcal{G}$. So $f \circ c$ is not differentiable at s_0. Thus $f \circ c$ is nowhere differentiable on (a, b).

Hence f is strongly nowhere differentiable, completing the proof. $\square$

4. Open questions

In the Main Theorem we test multivariant functions for differentiability by testing along each unit speed curve C^1 which is also $C^{1,\gamma}$ for some $\gamma \in (0,1]$. In terms of checking for differentiability, the extra smoothness assumption is not particularly natural. The following questions, which we state as conjectures, seem natural.

Conjecture 4.1. *There is a continuous function $f\colon \mathbb{R}^d \to \mathbb{R}$ whose restriction to any C^1 unit speed curve is nowhere differentiable.*

Conjecture 4.2. *Let U is a bounded open set in $\mathbb{R}^d$. The set of $C(\overline{U})$ functions whose restriction to each C^1 unit speed curve is nowhere differentiable contains a dense G_δ subset.*

If the first conjecture holds, it would be nice if the second one followed by arguments similar to the ones used to prove our Main Theorem. But our proof uses in an essential way the compactness result of Lemma 3.2. In our case the compactness follows from the Arzelà–Ascoli Theorem and the assumption the derivatives of the test curves satisfy a Hölder condition. If we are using as the set of test curves all C^1 unit speed curves, a direct adaptation of our methods would require that the set of all unit speed curves $c\colon [-1/n, 1/n] \to U$ is a countable union of compact sets in the C^1 topology, which is false. Thus, even if Conjecture 4.1 holds, new ideas are needed to prove Conjecture 4.2.

5. Appendix: Katzourakis' example

The first person to show a function having a property similar to being precisely $C^{0,\alpha}$ was Hardy in [6, Theorem 1.32] where he shows the Weierstrass function $W(x) = \sum_{n=1}^{\infty} a^n \cos(b^n \pi x)$ with $0 < a < 1$, $ab > 1$, and $\alpha = \ln(1/a)/\ln(b)$ satisfies the pointwise estimates

$$0 < \limsup_{y\to x} \frac{|W(x) - W(y)|}{|x-y|^\alpha} < \infty.$$

However his proof does not seem to show this inequality holds uniformly in x and thus does not show that W is α-Hölder on $\mathbb{R}$. Hardy's proof proceeds by extending W to a harmonic function on the upper half plane $\{x + iy : y \geq 0\}$ and using methods and results from harmonic analysis.

The first examples of precisely $C^{0,\alpha}$ functions were given by E. I. Berezhnoĭ [3]. He uses a modulus of continuity ω more general than the Hölder case $\omega(t) = t^\alpha$ and uses rather abstract methods from Banach space theory. In [8], Nikolaos I. Katzourakis constructs concrete examples of functions $K_{\alpha,\nu}$ that are precisely $C^{0,\alpha}$. We give a variant of his construction. We get a larger class of functions (we only require that b in the definition (5.2) is an even integer with $b^{1-\alpha} > 2$ while Katzourakis requires $b = 2^{2\nu}$ for a sufficiently large integer ν). Our elementary Lemma 5.3 lets us simplify the proof a bit by only having to look at difference quotients on intervals of the form $[b^{-m}j, b^{-m}(j+1)]$. However the main reason our construction is shorter is that Katzourakis proves his functions also satisfy the stronger "Hardy condition"

$$0 < \limsup_{y \to x} \frac{|K_{\alpha,\nu}(y) - K_{\alpha,\nu}(x)|}{|y-x|^{\alpha}} < \infty$$

for each $x \in \mathbb{R}$. We only show the less precise lower bound of (2.1). Proving Katzourakis' more precise result requires a substantial amount of extra work. We include this example for completeness and because it requires less machinery than any example we have seen in the literature.

Define a sawtooth function $\phi\colon \mathbb{R} \to \mathbb{R}$ by

$$\phi(x) := \operatorname{dist}(x, 2\mathbb{Z}).$$

Then ϕ is continuous, even, 2-periodic, and satisfies for each $x, y \in \mathbb{R}$

$$0 \le \phi(x) \le 1$$

and

$$|\phi(x) - \phi(y)| \le |x-y|. \tag{5.1}$$

For $0 < \alpha < 1$ and an even positive integer b, define

$$\Phi(x) := \sum_{k=0}^{\infty} b^{-k\alpha}\phi(b^k x). \tag{5.2}$$

Then Φ is continuous, even, 2-periodic, and thus bounded.

Theorem 5.1. *The function Φ is precisely $C^{0,\alpha}$ on $\mathbb{R}$ when $0 < \alpha < 1$, the number b is an even positive integer, and $b^{1-\alpha} > 2$.*

Theorem 5.1 follows from Proposition 5.2 and Proposition 5.4.

Proposition 5.2. *The function Φ is in $C^{0,\alpha}(\mathbb{R})$ when $0 < \alpha < 1$ and the b is an even positive integer.*

Proof. Since Φ is even and 2-periodic, it is enough to show Φ is Hölder continuous with exponent α on the interval $[0,1]$. Let $x, y \in [0,1]$. Without loss of generality we may assume $x \ne y$. Let m be a positive integer with

$$b^{-m} < |x-y| \le b^{-m+1}.$$

As ϕ takes its values in $[0,1]$ we have $|\phi(b^k x) - \phi(b^k y)| \le 1$. Using this, the inequality (5.1), and the triangle inequality

$$
\begin{aligned}
|\Phi(x) - \Phi(y)| &\le \sum_{k=0}^{m} b^{-k\alpha} |\phi(b^k x) - \phi(b^k y)| \\
&\qquad + \sum_{k=m+1}^{\infty} b^{-k\alpha} |\phi(b^k x) - \phi(b^k y)| \\
&\le \sum_{k=-\infty}^{m} b^{k(1-\alpha)} |x-y| + \sum_{k=m+1}^{\infty} b^{-k\alpha} \\
&= \frac{b^{m(1-\alpha)}}{1-b^{-(1-\alpha)}} |x-y| + \frac{b^{-(m+1)\alpha}}{1-b^{-\alpha}} \qquad (5.3) \\
&= \frac{b^{m(1-\alpha)}|x-y|^{1-\alpha}}{1-b^{-(1-\alpha)}} |x-y|^{\alpha} + \frac{b^{-m\alpha} b^{-\alpha}}{1-b^{-\alpha}}
\end{aligned}
$$

where the equality (5.3) comes from summing the two geometric series. As $|x-y| \le b^{-m+1}$

$$b^{m(1-\alpha)}|x-y|^{1-\alpha} \le b^{m(1-\alpha)}(b^{-m+1})^{1-\alpha} = b^{1-\alpha}.$$

The inequality $b^{-m} \le |x-y|$ implies

$$b^{-m\alpha} \le |x-y|^{\alpha}.$$

Using these inequalities in the estimate for $|\Phi(x) - \Phi(y)|$ gives

$$
\begin{aligned}
|\Phi(x) - \Phi(y)| &\le \frac{b^{1-\alpha}}{1-b^{-(1-\alpha)}} |x-y|^{\alpha} + \frac{b^{-\alpha}}{1-b^{-\alpha}} |x-y|^{\alpha} \\
&= C_\alpha |x-y|^{\alpha}
\end{aligned}
$$

where

$$C_\alpha = \frac{b^{1-\alpha}}{1-b^{-(1-\alpha)}} + \frac{b^{-\alpha}}{1-b^{-\alpha}} \ ,$$

which shows Φ is Hölder continuous on with exponent α on $[0,1]$. □

Lemma 5.3. *Let $f\colon \mathbb{R} \to \mathbb{R}$ and let $x \in \mathbb{R}$. Assume for some $\beta \in (0,1]$ there is a constant C with*

$$|f(x) - f(y)| \le C|x-y|^{\beta}$$

for all $y \in \mathbb{R}$. Let $x_0 \le x \le x_1$. Then

$$|f(x_1) - f(x_0)| \le 2C|x_1 - x_0|^{\beta}.$$

Proof.

$$
\begin{aligned}
|f(x_1) - f(x_0)| &\le |f(x_1) - f(x)| + |f(x) - f(x_0)| \\
&\le C|x_1 - x|^{\beta} + C|x - x_0|^{\beta} \le 2C|x_1 - x_0|^{\beta}.
\end{aligned}
$$

$\square$

Proposition 5.4. *Let Φ be as in (5.2) so $0 < \alpha < 1$ and the b is an even positive integer. Assume furthermore that $b^{1-\alpha} > 2$. Let $\beta > \alpha$. Then for each $x \in \mathbb{R}$*

$$\limsup_{y \to x} \frac{|\Phi(x) - \Phi(y)|}{|x-y|^\beta} = \infty.$$

Proof. Let m be a positive integer. Let j and k be nonnegative integers. As ϕ is periodic with period 2 when $k > m$ the power b^{k-m} is an even integer (as b is even) and so

$$\phi(b^k(b^{-m}(j+1))) - \phi(b^k(b^{-m}j)) = \phi(b^{k-m}j + b^{k-m}) - \phi(b^{k-m}j) = 0.$$

Therefore the sum for $\Phi(b^{-m}(j+1)) - \Phi(b^{-m}j)$ reduces to a finite sum

$$\begin{aligned}
&\Phi(b^{-m}(j+1)) - \Phi(b^{-m}j) \\
&\qquad = \sum_{k=0}^{m} b^{-k\alpha} \left[\phi(b^k(b^{-m}(j+1))) - \phi(b^k(b^{-m}j))\right] \\
&\qquad = b^{-m\alpha}\left[\phi(j+1) - \phi(j)\right] \\
&\qquad\qquad + \sum_{k=0}^{m-1} b^{-k\alpha}\left[\phi(b^{k-m}j + b^{k-m}) - \phi(b^{k-m}j)\right].
\end{aligned}$$

But $\phi(j+1) - \phi(j) = \pm 1$ for all j. As ϕ is Lipschitz with Lipschitz constant one, $|\phi(b^{k-m}j + b^{k-m}) - \phi(b^{k-m}j)| \le b^{k-m}$. Putting this together with reverse triangle inequalitygives

$$\begin{aligned}
&|\Phi(b^{-m}(j+1)) - \Phi(b^{-m}j)| \\
&\qquad \ge b^{-\alpha m} - \sum_{k=0}^{m-1} b^{-\alpha k}|\phi(b^{k-m}j + b^{k-m}) - \phi(b^{k-m}j)| \\
&\qquad \ge b^{-\alpha m} - \sum_{k=0}^{m-1} b^{-k\alpha} b^{k-m} \\
&\qquad = b^{-\alpha m} - b^{-m}\sum_{k=0}^{m-1} b^{k(1-\alpha)} \\
&\qquad = b^{-\alpha m} - b^{-m}\frac{1 - b^{m(1-\alpha)}}{1 - b^{1-\alpha}} \\
&\qquad = b^{-\alpha m}\left(1 - b^{-m+\alpha m}\,\frac{1 - b^{m(1-\alpha)}}{1 - b^{1-\alpha}}\right)
\end{aligned}$$

$$
\begin{aligned}
&= b^{-\alpha m}\left(1 - b^{-m(1-\alpha)}\,\frac{b^{m(1-\alpha)}-1}{b^{1-\alpha}-1}\right)\\
&= b^{-\alpha m}\left(1 - \frac{1-b^{-m(1-\alpha)}}{b^{1-\alpha}-1}\right)\\
&\geq b^{-\alpha m}\left(1 - \frac{1}{b^{1-\alpha}-1}\right).
\end{aligned}
$$

Thus

$$
|\Phi(b^{-m}(j+1)) - \Phi(b^{-m}j)| \geq M_\alpha b^{-\alpha m} \tag{5.4}
$$

where

$$
M_\alpha = 1 - \frac{1}{b^{1-\alpha}-1}.
$$

When $b^{1-\alpha} > 2$

$$
M_\alpha > 0.
$$

Towards a contradiction assume there is a $x \in \mathbb{R}$ with

$$
\limsup_{y\to x} \frac{|\Phi(y)-\Phi(x)|}{|y-x|^\beta} < \infty.
$$

By the periodicity of Φ we can assume $x \geq 0$. As Φ is bounded this implies there is a $C > 0$ so that

$$
|\Phi(y) - \Phi(x)| \leq C|x-y|^\beta
$$

holds for all $y \in \mathbb{R}$. For each positive integer m choose a nonnegative integer j such that

$$
b^{-m}j \leq x < b^{-m}(j+1).
$$

Letting $x_0 = b^{-m}j$ and $x_1 = b^{-m}(j+1)$ in Lemma 5.3

$$
|\Phi(x_1) - \Phi(x_0)| \leq 2C|x_1 - x_0|^\beta = 2Cb^{-\beta m}.
$$

Combining this with the inequality (5.4) gives

$$
M_\alpha b^{-\alpha m} \leq |\Phi(x_1) - \Phi(x_0)| \leq 2Cb^{-\beta m},
$$

which implies

$$
M_\alpha b^{m(\beta-\alpha)} \leq 2C.
$$

Using $M_\alpha > 0$ and $\beta - \alpha > 0$, letting $m \to \infty$ gives the required contradiction. $\square$

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